

THE MÖBIUS TRANSFORMATION OF CONTINUED FRACTIONS WITH BOUNDED UPPER AND LOWER PARTIAL QUOTIENTS

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ABSTRACT. Let $h: x \mapsto \frac{ax+b}{cx+d}$ be the nondegenerate Möbius transformation with integer entries. We get a bound of the continued fraction of $h(x)$ by the upper and lower bound of continued fraction of x , which extends a result of Stambul [7].

1. INTRODUCTION

A continued fraction representation of a number $x \in \mathbb{R}$ is an expansion of the form

$$(1) \quad x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots}}}}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}^+$, $i = 1, 2, \dots$. A continued fraction may be finite or infinite. If (1) is a finite continued fraction, we denote it by $[a_0; a_1, a_2, \dots, a_n]$; if (1) is infinite, then we denote it by $[a_0; a_1, a_2, \dots]$. We call a_j the j th partial quotient. It is a well known fact that the continued fraction of x is infinite iff x is irrational.

Given a nondegenerate 2×2 matrix M with integer entries, that is $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and the determinant $ad - bc \neq 0$, we can define the associated Möbius transformation $h: x \mapsto \frac{ax+b}{cx+d}$. We also denote by

$$h(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

In this paper, we study the bound of partial quotients under the Möbius transformation. We will use $[x] = \max\{j \in \mathbb{Z} : j \leq x\}$. Our main result is

Theorem 1.1. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a nondegenerate matrix with entries in \mathbb{Z} and h be the associated Möbius transformation. Let $x = [a_0; a_1, a_2, \dots]$ be a real number such that $B_1 \leq a_j \leq B_2$ for j large enough. Let $h(x) = [a_0^*; a_1^*, a_2^*, \dots]$. Then $a_j^* \leq \lfloor \frac{D-1}{B_1} \rfloor + \lfloor D \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4B_1 B_2}}{2B_1} \rfloor$ for large j , where $D = |\det(M)|$.*

Now we always assume $x = [a_0; a_1, a_2, \dots]$ and $\frac{ax+b}{cx+d} = [a_0^*; a_1^*, a_2^*, \dots]$ with $D = |ad - bc| \geq$

1. Set $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $h(x) = \frac{ax+b}{cx+d}$.

It is an old result that a real number $\frac{ax+b}{cx+d}$ has bounded partial quotients if x does [2, 5, 6], so the quantitative bound becomes an interesting question. Lagarias-Shallit [3] and Cusick-France [1] obtained a quantitative bound, which stated that if x has bounded partial quotients with $a_j \leq K$ eventually, then the associated partial quotients a_j^* of $\frac{ax+b}{cx+d}$ satisfy $a_j^* \leq D(K+2)$ eventually.

Using an algorithm developed by Liardet-Stambul [4] to calculate the partial quotients of $h(x)$, Stambul gave an upper bound $a_j^* \leq D - 1 + \lfloor D \frac{K + \sqrt{K^2 + 4K}}{2} \rfloor$ [7], which is the $B_1 = 1$ case of Theorem 1.1. In this paper, we concern the partial quotients with lower and upper bound at the same time. Our methods are based on the refining of analysis in papers [4, 7].

2. ALGORITHM FOR PARTIAL QUOTIENTS

In this section, we will introduce some notations and the algorithm developed by Liardet-Stambul [4, 7] to calculate the partial quotients of $h(x)$. Let $M_{2,\mathbb{N}}$ be the set of all matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ($a, b, c, d \in \mathbb{N}$) such that $ad - bc \neq 0$. M is said to be in \mathcal{D}_2 when $a \geq c$ and $b \geq d$, in \mathcal{D}'_2 when $a \leq c$ and $b \leq d$, and in ε_2 when $(a - c)(b - d) < 0$. $\{\mathcal{D}_2, \mathcal{D}'_2, \varepsilon_2\}$ is a partition of $M_{2,\mathbb{N}}$.

It is easy to see that $M \in \varepsilon_2$ satisfies

$$(2) \quad \max\{|a| + |b|, |c| + |d|\} \leq |\det M| = D.$$

For all matrices $M \in \mathcal{D}_2 \cup \mathcal{D}'_2$, there exists a unique factorization

$$(3) \quad M = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} M'$$

such that $c_0 \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{N}^+$ and $M' \in \varepsilon_2$ [4]. This factorization will be denoted by $M = \Pi_{c_0 c_1 \dots c_n} M'$. Moreover, $[c_0; c_1, c_2, \dots, c_{n-1}]$ is the common sequence of partial quotients of $\frac{a}{c}$ and $\frac{b}{d}$ if $n \neq 1$. c_n can be determined by the following several cases [4].

Case 1 : If $\frac{a}{c} = [c_0; c_1, c_2, \dots, c_{n-1}]$, then c_n is the n th partial quotient of $\frac{b}{d}$.

Case 2 : If $\frac{b}{d} = [c_0; c_1, c_2, \dots, c_{n-1}]$, then c_n is the n th partial quotient of $\frac{a}{c}$.

Case 3 : Otherwise, c_n is the smaller one of n th partial quotients of $\frac{a}{c}$ and $\frac{b}{d}$.

Assume $M \in \varepsilon_2$ and h is the associated Möbius transformation. Let $x = [a_0; a_1, a_2, \dots] > 1$. Recall the algorithm in [4, 7] to compute the partial quotients of $h(x)$.

Step 0: $M_0 = M \in \varepsilon_2, j = 0, n = 0$.

Let j_1 be the smallest positive integer (see [4] for the existence) such that $M_0 \Pi_{a_0 a_1 \dots a_{j_1-1}} \in \varepsilon_2$ and $M_0 \Pi_{a_0 a_1 \dots a_{j_1}} \in \mathcal{D}_2 \cup \mathcal{D}'_2$. Factorizing $M_0 \Pi_{a_0 a_1 \dots a_{j_1}}$ as (3), we get

$$(\text{Output-0}) \quad M_0 \Pi_{a_0 a_1 \dots a_{j_1}} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n_1} & 1 \\ 1 & 0 \end{pmatrix} M_1$$

with $M_1 \in \varepsilon_2$.

Step 1: $M_1 \in \varepsilon_2, j = j_1 + 1, n = n_1 + 1$.

Let $j_2 \geq j_1 + 1$ be the smallest positive integer such that $M_1 \Pi_{a_{j_1+1} a_{j_1+2} \dots a_{j_2-1}} \in \varepsilon_2$ and $M_1 \Pi_{a_{j_1+1} a_{j_1+2} \dots a_{j_2}} \in \mathcal{D}_2 \cup \mathcal{D}'_2$. Factorizing $M_1 \Pi_{a_{j_1+1} a_{j_1+2} \dots a_{j_2}}$ as (3), we get

$$(\text{Output-1}) \quad M_1 \Pi_{a_{j_1+1} a_{j_1+2} \dots a_{j_2}} = \begin{pmatrix} c_{n_1+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n_1+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n_2} & 1 \\ 1 & 0 \end{pmatrix} M_2$$

with $M_2 \in \varepsilon_2$.

Step 2: $M_2 \in \varepsilon_2, j = j_2 + 1, n = n_2 + 1$.

Let $j_3 \geq j_2 + 1$ be the smallest positive integer such that $M_2 \Pi_{a_{j_2+1} a_{j_2+2} \dots a_{j_3-1}} \in \varepsilon_2$ and $M_2 \Pi_{a_{j_2+1} a_{j_2+2} \dots a_{j_3}} \in \mathcal{D}_2 \cup \mathcal{D}'_2$. Factorizing $M_2 \Pi_{a_{j_2+1} a_{j_2+2} \dots a_{j_3}}$ as (3), we get

$$(\text{Output-2}) \quad M_2 \Pi_{a_{j_2+1} a_{j_2+2} \dots a_{j_3}} = \begin{pmatrix} c_{n_2+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n_2+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n_3} & 1 \\ 1 & 0 \end{pmatrix} M_3$$

with $M_3 \in \varepsilon_2$.

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Step k: $M_k \in \varepsilon_2, j = j_k + 1, n = n_k + 1$.

Let $j_{k+1} \geq j_k + 1$ be the smallest positive integer such that $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}-1}} \in \varepsilon_2$ and $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}}} \in \mathcal{D}_2 \cup \mathcal{D}'_2$. Factorizing $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}}}$ as (3), we get

$$(\text{Output-k}) \quad M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}}} = \begin{pmatrix} c_{n_k+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{n_k+2} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n_{k+1}} & 1 \\ 1 & 0 \end{pmatrix} M_{k+1}$$

with $M_{k+1} \in \varepsilon_2$.

Putting all the Output (Output-k) together, we get a sequence

$$(\text{Alloutput-k}) \quad c_0 c_1 c_2 c_3 \cdots c_{n_k}$$

Unfortunately, many c_i maybe zero, thus we must introduce the contraction map μ . For any word $c_0 c_1 c_2 c_3 \cdots c_n \in \mathbb{N}^n$, let μ be the contraction map which transforms a word into a word where all letters are positive integers (except perhaps the first one), replacing from left to right factors $a0b$ by the letter $a+b$.

By the fact

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix},$$

we have

$$(4) \quad \Pi_{\mu(c_0 c_1 c_2 c_3 \cdots c_n)} = \Pi_{c_0 c_1 c_2 c_3 \cdots c_n}$$

Let μ act on (Alloutput-k), then we get

$$(\text{Partialquotients}) \quad c_0^* c_1^* c_2^* c_3^* \cdots c_{n'_k}^* = \mu(c_0 c_1 c_2 c_3 \cdots c_{n_k}).$$

By the arguments in [4], n'_k goes to infinity as k does, moreover,

$$(5) \quad \frac{ax+b}{cx+d} = [c_0^*, c_1^*, \dots, c_{n'_k-1}^*, \dots]$$

and the n'_k th partial quotient following $c_{n'_k-1}^*$ is no less than $c_{n'_k}^*$.

Now, we give a quantitative estimate about c_i in (Alloutput-k).

Lemma 2.1. Assume $M \in \varepsilon_2$ and $x = [a_0; a_1, a_2, \dots] > 1$. Let h be the associated Möbius transformation and $D = |\det M| \geq 1$. Suppose $a_j \leq K$ for some $K \in \mathbb{N}^+$. We do the algorithm as above, then the following three claims hold,

- (i): For any $n_k < j \leq n_{k+1} - 1$, $c_j \leq D - 1$
- (ii): For any k , $c_{n_{k+1}} \leq DK$
- (iii): If for some k , $c_{n_{k+1}} \geq D$, then the right upper entry of M_{k+1} must be zero, that is M_{k+1} has the form

$$(6) \quad M_{k+1} = \begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix}$$

Proof. The three claims are from [7]. We rewrite the proof here to make the paper more readable. By the algorithm, we already have $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}-1}} \in \varepsilon_2$ and $M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}}} \in \mathcal{D}_2 \cup \mathcal{D}'_2$.

For simplicity, let $M' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M_k \Pi_{a_{j_k+1} a_{j_k+2} \cdots a_{j_{k+1}-1}} \in \varepsilon_2$ and $f = a_{j_{k+1}} \leq K$. Then $M' \Pi_f \in \mathcal{D}_2 \cup \mathcal{D}'_2$.

If $\gamma = 0$, then

$$M'\Pi_f = \begin{pmatrix} \alpha f + \beta & \alpha \\ \delta & 0 \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}'_2$$

and we must have $\alpha f + \beta \geq \delta$. Thus

$$M'\Pi_f = \begin{pmatrix} \alpha f + \beta & \alpha \\ \delta & 0 \end{pmatrix} = \begin{pmatrix} \lfloor \frac{\alpha f + \beta}{\delta} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ (\alpha f + \beta) \bmod \delta & \alpha \end{pmatrix}.$$

In this case, in order to prove the Lemma, it suffices to show that

$$(7) \quad \lfloor \frac{\alpha f + \beta}{\delta} \rfloor \leq DK.$$

Otherwise, one has

$$(8) \quad DK + 1 \leq \lfloor \frac{\alpha f + \beta}{\delta} \rfloor = \lfloor \frac{\alpha f}{\delta} + \frac{\beta}{\delta} \rfloor \leq \lfloor \frac{\alpha K}{\delta} + \frac{\beta}{\delta} \rfloor,$$

since $f \leq K$.

By the fact $M' = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in \varepsilon_2$, we have $\beta < \delta$, $|\alpha| + |\beta| \leq D$. This is contradicted to (8).

If $\alpha = 0$, then

$$M'\Pi_f = \begin{pmatrix} \beta & 0 \\ \gamma f + \delta & \gamma \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}'_2$$

and we must have $\gamma f + \delta \geq \beta$. Thus

$$M'\Pi_f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lfloor \frac{\gamma f + \delta}{\beta} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ (\gamma f + \delta) \bmod \delta & \gamma \end{pmatrix}.$$

In this case, we can still prove the Lemma like the case $\gamma = 0$.

If $\alpha, \gamma \geq 1$, then

$$M'\Pi_f = \begin{pmatrix} \alpha f + \beta & \alpha \\ \gamma f + \delta & \gamma \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}'_2.$$

By the algorithm, $n_k \leq j \leq n_{k+1} - 1$, c_j is the common partial quotient of $\frac{\alpha}{\gamma}$ and $\frac{\alpha f + \beta}{\gamma f + \delta}$.

We first show claim 1 holds. Indeed, $\alpha \leq D$ and $\gamma \geq 1$. If $\alpha = D$ and $\gamma = 1$, we must have $\beta = 0$ and $\delta = 1$. This implies claim 1 when we consider the partial quotient of $\frac{\alpha f + \beta}{\gamma f + \delta}$. Otherwise ($\alpha = D$ and $\gamma = 1$ do not hold) claim 1 holds if we consider the partial quotient of $\frac{\alpha}{\gamma}$.

Suppose the last letter, i.e. $c_{n_{k+1}} \geq D$, then we must have $\frac{\alpha}{\gamma} = [c_{j_k+1}; c_{j_k+2}, c_{j_k+2}, \dots, c_{j_{k+1}-1}]$ by the (Case1-Case3) and $c_{n_{k+1}} \geq D$ is the $n_{k+1} - n_k + 1$ th partial quotient of $\frac{\alpha f + \beta}{\gamma f + \delta}$. This implies claims 2 and 3 if we can show

$$\frac{1}{DK} \leq \frac{\alpha f + \beta}{\gamma f + \delta} \leq DK.$$

We only prove the fact $\frac{\alpha f + \beta}{\gamma f + \delta} \leq DK$, the proof of lower bound $\frac{1}{DK} \leq \frac{\alpha f + \beta}{\gamma f + \delta}$ is the same.

If $\gamma f + \delta \geq 2$, then $\frac{\alpha f + \beta}{\gamma f + \delta} \leq \frac{DK + D}{2} \leq DK$. If $\gamma f + \delta \leq 1$, then we have $\delta = 0$ and $\gamma = K = 1$. This implies $\beta = D$ and $\alpha = 0$. We still have $\frac{\alpha f + \beta}{\gamma f + \delta} \leq DK$.

□

3. SOME LEMMAS

We say a Möbius transformation $h(\cdot) = M \cdot$ can not change the continued fraction eventually, if for any x , there exists some $N \in \mathbb{N}$ such that the n th partial quotients of $h(x)$ and x are the same for any $n \geq N$.

Lemma 3.1. *The following forms of Möbius transformations can not change the continued fraction eventually,*

$$(9) \quad S = \left\{ \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} k_2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k_3 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

where $k_1, k_2, k_3 \in \mathbb{Z}$.

Proof. The proof is based on direct computation. \square

Remark: The determinant of each matrix in S is ± 1 .

Lemma 3.2. *Assume $a, b, c, d \in \mathbb{Z}$ and $ad - bc \neq 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be rewritten in the following form*

$$(10) \quad M = S_1 S_2 \cdots S_n M'$$

with $M' \in \varepsilon_2$. Moreover if $D = \det M = 1$, then M' can be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Using Möbius transformation $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in S$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in S$, we can assume $a, c \geq 0$.

Using Möbius transformation $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \in S$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S$, M can be changed to $M_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$ with $a_1 \geq 1$.

Using Möbius transformation $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in S$ and $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in S$, M_1 can be changed to $M' = \begin{pmatrix} a_1 & b_1 \bmod |d_1| \\ 0 & |d_1| \end{pmatrix} \in \varepsilon_2$.

Moreover, if $D = 1$, we must have $a_1 = 1, |b_1| = 1$ and $b_1 \bmod |d_1| = 0$. \square

Remark: If $|\det M| = 1$, then the associated Möbius transformations can not change the continued fraction eventually.

Lemma 3.3. *Let $M \in \varepsilon_2$ and $D = |\det M| \geq 2$. Let $x = [a_0; a_1, a_2, \dots]$ such that $B_1 \leq a_j \leq B_2$ for all $j \geq 0$. Using the Algorithm in section 2, we get a sequence $c_0^* c_1^* c_2^* c_3^* \cdots$ by (Partialquotients). If $c_0^* = 0$, then*

$$(11) \quad c_1^* \leq \lfloor D y_0 \rfloor$$

where $y_0 = [B_2; B_1, B_2, B_1, \dots] \triangleq [\overline{B_2, B_1}] = \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_1}$. Moreover, the equality in (11) holds iff $a = 0, b = 1, c = D$ and $d = 0$.

In addition, assume $M \neq \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, then

$$(12) \quad c_1^* \leq \max\left\{\left\lfloor \frac{D}{4} y_0 + 1 \right\rfloor, D - 1\right\}$$

if $c_0^* = 0$.

Proof. Let

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, c_n],$$

then

$$\Pi_{a_0 a_1 \dots a_n} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Thus we have the following simple facts

$$(13) \quad M \Pi_{a_0 a_1 \dots a_n} = \begin{pmatrix} ap_n + bq_n & ap_{n-1} + bq_{n-1} \\ cp_n + dq_n & cp_{n-1} + dq_{n-1} \end{pmatrix},$$

and

$$\lim_{n \rightarrow \infty} \frac{ap_n + bq_n}{cp_n + dq_n} = \frac{ap_{n-1} + bq_{n-1}}{cp_{n-1} + dq_{n-1}} = \frac{ax + b}{cx + d}.$$

If $c_0^* = 0$, then c_1^* is the second common partial quotient of $\frac{ap_n + bq_n}{cp_n + dq_n}$ and $\frac{ap_{n-1} + bq_{n-1}}{cp_{n-1} + dq_{n-1}}$ for any large n . Combining with (13), we must have

$$(14) \quad c_1^* = \lfloor \frac{cx + d}{ax + b} \rfloor.$$

Now we are in a position to prove the Lemma, based on (14).

Case 1: $a \geq 1$

Using $x > 1$, one has

$$\begin{aligned} \frac{cx + d}{ax + b} &\leq \frac{cx + d}{ax} \\ &< \frac{c + d}{a} \\ &\leq D \end{aligned}$$

where the third inequality holds by (2). This implies $c_1^* \leq D - 1$.

Case 2: $a = 0$

In this case, we have $b > d$, $bc = D$ and $c + d \leq D$ by $M \in \varepsilon_2$, and

$$(15) \quad c_1^* = \lfloor \frac{D}{b^2}x + \frac{d}{b} \rfloor.$$

If $b \geq 2$, by (15), one has

$$c_1^* \leq \lfloor \frac{D}{4}x + 1 \rfloor.$$

Notice that if a real number with bounded partial quotients in $[B_1, B_2] \cap \mathbb{Z}$ is such that $x \leq y_0$, then

$$c_1^* \leq \lfloor \frac{D}{4}y_0 + 1 \rfloor \leq \lfloor Dy_0 \rfloor - 1,$$

since $y_0 \geq \frac{\sqrt{5}+1}{2}$ and $D \geq 2$.

If $b = 1$, we must have $c = D$ and $d = 0$.

Putting all the cases together, we complete the proof. \square

Lemma 3.4. *Let $M \in \varepsilon_2$ with the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ and $D = |\det M| \geq 1$. Let $x = [a_0; a_1, a_2, \dots]$ such that $B_1 \leq a_j \leq B_2$ for all $j \geq 0$. Applying the Algorithm in section 2 to $M \cdot x$, we get a sequence $c_0^* c_1^* c_2^* c_3^* \dots$ by (Partialquotients). If $c_0^* = 0$, we must have*

$$c_1^* \leq \lfloor \frac{D}{x_0} \rfloor,$$

where $x_0 = [B_1; B_2, B_1, B_2, \dots] \triangleq \overline{[B_1, B_2]} = \frac{B_2 B_1 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_2}$.

Proof. Let $b = 0$ in (14), then we get

$$(16) \quad c_1^* = \lfloor \frac{cx + d}{ax} \rfloor.$$

Notice that if a real number with bounded partial quotients in $[B_1, B_2] \cap \mathbb{Z}$ is such that $x \geq x_0$, then

$$(17) \quad c_1^* \leq \lfloor \frac{cx_0 + d}{ax_0} \rfloor.$$

Thus in order to prove this Lemma, it suffices to show

$$(18) \quad \frac{cx_0 + d}{ax_0} \leq \frac{D}{x_0}.$$

If $a = 1$, we must have $c = 0$ and $d = D$, this implies (18).

If $a \geq 2$, we already have $ad = D$ and $c \leq a - 1$.

Case 1: $D \geq 2x_0 > 2$

One has

$$\begin{aligned} cx_0 + d &\leq (a - 1)x_0 + \frac{D}{2} \\ &\leq \frac{D(a - 1)}{2} + \frac{D}{2} \\ &\leq Da \end{aligned}$$

This implies (18).

Case 2: $x_0 \leq D < 2x_0$

It suffices to show

$$(19) \quad \frac{cx_0 + d}{ax_0} < 2.$$

This is obvious by the following computation,

$$\begin{aligned} cx_0 + d &\leq (a - 1)x_0 + D \\ &< ax_0 + 2x_0 \\ &\leq 2ax_0 \end{aligned}$$

This implies (23).

Case 3: $D < x_0$

By direct computation,

$$\begin{aligned} \frac{cx_0 + d}{ax_0} &= \frac{c}{a} + \frac{D}{a^2 x_0} \\ &< \frac{a - 1}{a} + \frac{1}{a^2} \\ &< 1. \end{aligned}$$

This also implies (18).

□

4. PROOF OF THEOREM 1.1

Proof of Theorem 1.1:

Proof. Suppose $x = [a_0; a_1, a_2, \dots]$ is such that $B_1 \leq a_j \leq B_2$ for $j \geq j_0$, and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such that $D = |\det M| \geq 1$. By Lemmas 3.1 and 3.2, we may assume $M \in \varepsilon_2$. By the fact

$$(20) \quad h(x) = M \cdot x = M \Pi_{a_0 a_1 \dots a_{j_0}} \cdot [a_{j_0+1}; a_{j_0+2}, \dots]$$

combining with (3), in order to prove Theorem 1.1, we only need to prove the case when all the partial quotients of x satisfy $B_1 \leq a_i \leq B_2$.

By the Algorithm, it suffices to show that for any word $k_1 0 k_2 0 \dots 0 k_p$ in (Alloutput-k) with $k_i \in \mathbb{N}^+, i = 1, 2, \dots, p$, we have

$$(21) \quad k_1 + k_2 + \dots + k_p \leq \lfloor \frac{D-1}{B_1} \rfloor + \lfloor D \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_1} \rfloor.$$

Assume k_1 is the last letter of k th step (Alloutput-k). Then the output of $k+1$ th step is $0 k_2$, $k+2$ th step is $0 k_3$, \dots .

Case 1: $k_1 \geq D$

By (iii) of Lemma 2.1, M_{k+1} has the form

$$M_{k+1} = \begin{pmatrix} a_k & 0 \\ c_k & d_k \end{pmatrix} \in \varepsilon_2.$$

By Lemma 3.4, we have

$$\sum_{j=2}^p k_j \leq \lfloor \frac{D}{x_0} \rfloor.$$

By (ii) of Lemma 2.1, $k_1 \leq D B_2$, then

$$\begin{aligned} \sum_{j=1}^p k_j &\leq \lfloor \frac{D}{x_0} \rfloor + D B_2 \\ &\leq \lfloor D \frac{B_2 B_1 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_1} \rfloor \\ &\leq \lfloor \frac{D-1}{B_1} \rfloor + \lfloor D \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_1} \rfloor. \end{aligned}$$

This implies the Theorem in this case.

By the Remark following Lemma 3.2, we can assume $D \geq 2$.

Case 2: $k_1 \leq D-1$

If $M_{k+1} \neq \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, by (13) one has

$$\sum_{j=2}^p k_j \leq \max\{\lfloor \frac{D}{4} y_0 + 1 \rfloor, D-1\}.$$

Direct computation (splitting the computation into $B_1 = 1$ or $B_1 \geq 2$),

$$\begin{aligned} \sum_{j=1}^p k_j &\leq D-1 + \max\{\lfloor \frac{D}{4} y_0 + 1 \rfloor, D-1\} \\ &\leq \lfloor \frac{D-1}{B_1} \rfloor + \lfloor D \frac{B_1 B_2 + \sqrt{B_1^2 B_2^2 + 4 B_1 B_2}}{2 B_1} \rfloor. \end{aligned}$$

This implies the Theorem in this case.

If $M_{k+1} = \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$, by (21) one has

$$c_1^* \leq \lfloor Dy_0 \rfloor.$$

Thus in order to prove the Theorem in this case, it suffices to show

$$(22) \quad k_1 \leq \frac{D-1}{B_1}.$$

By the Algorithm of k th step, we have

$$(23) \quad M_k \Pi_{a_1 a_2 \dots a_N} = \Pi_{c_1 c_2 \dots c_{N'-1}} \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \in \mathcal{D}_2 \cup \mathcal{D}'_2,$$

and $M_k \Pi_{a_1 a_2 \dots a_{N-1}} \in \varepsilon_2$.

This implies

$$(24) \quad M_k \Pi_{a_1 a_2 \dots a_{N-1}} = \Pi_{c_1 c_2 \dots c_{N'-1}} \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix} \begin{pmatrix} a_N & 1 \\ 1 & 0 \end{pmatrix}^{-1}.$$

By direct computation, one has

$$(25) \quad M_k \Pi_{a_1 a_2 \dots a_{N-1}} = \Pi_{c_1 c_2 \dots c_{N'-1}} \begin{pmatrix} k_1 & -k_1 a_N + D \\ 1 & -a_N \end{pmatrix}.$$

Since all entries of $M_k \Pi_{a_1 a_2 \dots a_{N-1}}$ are non-negative, we must have

$$(26) \quad -k_1 a_N + D \geq 1.$$

This implies

$$k_1 \leq \lfloor \frac{D-1}{B_1} \rfloor,$$

since $a_N \geq B_1$. We complete the proof. □

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